Applications

1. LOANS **(** Derive formulas for payments and outstanding balances)

Recall

$$\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}^n \begin{pmatrix} L \\ x \end{pmatrix} = \begin{pmatrix} f(L, r, n, x) \\ x \end{pmatrix} = \begin{pmatrix} balance\ after\ n\ payments \\ x \end{pmatrix}$$

For simplicity, we let $i = \dfrac{r}{12}$ in the matrix $\begin{bmatrix} 1+\frac{r}{12} & -1 \\ 0 & 1 \end{bmatrix}$. To compute its n^{th} power

we must find eigenvalues and eigenvectors.

The characteristic polynomial of the loan matrix $\begin{bmatrix} 1+i & -1 \\ 0 & 1 \end{bmatrix}$ is

$P(\lambda) = \det \begin{bmatrix} (1+i)-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix} = \left[(1+i)-\lambda\right](1-\lambda) = 0$, and thus $\lambda_1 = 1+i$

and $\lambda_2 = 1$. I leave it as an exercise for you to show that representative eigenvectors for

the loan matrix are $\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}^{(2)} = \begin{pmatrix} \frac{1}{i} \\ 1 \end{pmatrix}$.

Using the equation $A^n = P\,D^n\,P^{-1}$,

$$\begin{bmatrix} 1+i & -1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & \frac{1}{i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1+i)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{i} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)^n & \frac{1}{i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{i} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)^n & -(1+i)^n \frac{1}{i} + \frac{1}{i} \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1+i & -1 \\ 0 & 1 \end{bmatrix}^n \begin{pmatrix} L \\ x \end{pmatrix} = \begin{bmatrix} (1+i)^n & -(1+i)^n \frac{1}{i} + \frac{1}{i} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} L \\ x \end{pmatrix}$$

$$= \begin{pmatrix} L(1+i)^n - \left((1+i)^n \frac{1}{i} - \frac{1}{i} \right) x \\ x \end{pmatrix}$$

$$= \begin{pmatrix} L(1+i)^n - \frac{x}{i}\left[(1+i)^n - 1 \right] \\ x \end{pmatrix}$$

Formula # 1

The outstanding balance, B, of a loan after n payments have been made is

$$B = L(1+i)^n - \frac{x}{i}\left[(1+i)^n - 1 \right]$$

Formula # 2

A loan is completely paid if the outstanding balance

$$L(1+i)^n - \frac{x}{i}\left((1+i)^n - 1 \right) = 0$$

If we solve for x we can determine the monthly payment.

$$\frac{x}{i}\left((1+i)^{n}-1\right)=L(1+i)^{n}$$

$$x=\frac{L(1+i)^{n}\cdot i}{(1+i)^{n}-1}$$

which simplifies to

$$\boxed{x=\frac{L\cdot i}{1-(1+i)^{-n}}}$$

Homework

Redo problems involving loans (Linear Transformations PDF) using the two formulas from the previous page.

Networks Probabilities and Matrices

Stochastic Vectors

The vector $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$ is called **stochastic** (or **probability** vector) if

a. $0 \le p_i \le 1$

b. $\sum_{i=1}^{n} p_i = 1$

Examples

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \quad \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{pmatrix}, \quad \begin{pmatrix} 0.0001 \\ 0.9 \\ 0.09 \\ 0.009 \\ 0.0009 \end{pmatrix}$$

Converting regular vectors into stochastic

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be any vector with either nonnegative or nonpositive entries. Then

the vector $\frac{1}{\sum_{i=1}^{n} v_i} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is stochastic. $\left(\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \to \frac{1}{10} \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} \\ \frac{3}{10} \\ \frac{6}{10} \end{pmatrix} \right)$

Stochastic Matrices

A matrix S is called stochastic if all its columns are probability vectors (column stochastic or left stochastic).

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.25 & 0.7 & 0 \\ 0.5 & 0.1 & 0.6 \\ 0.25 & 0.2 & 0.4 \end{bmatrix} \qquad \begin{bmatrix} 0.9 & 0.40 & 0 & 0 \\ 0.05 & 0.40 & 0 & 0 \\ 0.03 & 0.10 & 0.80 & 0.99 \\ 0.02 & 0.10 & 0.20 & 0.01 \end{bmatrix}$$

Theorem 9

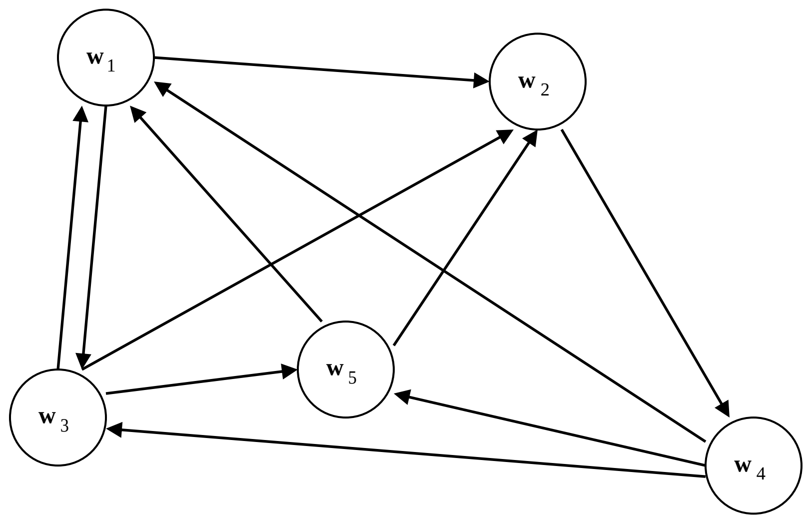
If S is an $n \times n$ stochastic matrix, then it has a dominant (largest) eigenvalue $\lambda_1 = 1$ and the rest of its eigenvalues

$$|\lambda_i| \le 1 \qquad\qquad (i = 2, 3, \ldots, n)$$

Proof Beyond the level of this course

2. PageRank

Consider the following group of webpages with the given links.



a. Find the **adjacency** matrix (A) for the above graph if

$$a_{ij} = \begin{cases} 1 & \text{if there is a direct link from } j \text{ to } i \\ 0 & \text{if there is no direct link from } j \text{ to } i \end{cases}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

b. Convert the adjacency matrix (A) into a stochastic matrix (S).

$$S = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

c. Define the "Google Matrix", G, of the given network. For a general $n \times n$ stochastic matrix S, we define

$$G = p \cdot S + \frac{(1-p)}{n} \cdot \left[1\right]_{n \times n}$$

where the probability p is called a dumping factor ($p \simeq 0.85$ these days). What this essentially says is that with probability 0.85 a random surfer uses links to navigate from one webpage to another, and with probability 0.15 types a URL (web address) or uses bookmarks to continue surfing the web.

$$G = 0.85 \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} + \frac{0.15}{5} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d. Rank the five webpages using the eigenvector corresponding to the dominant eigenvalue of the matrix G.

Transition probabilities, to be explained in class, after 2, 10 and 1000 iterations are given below.

$$G^2 = \begin{bmatrix} 0.180 & 0.301 & 0.180 & 0.260 & 0.060 \\ 0.184 & 0.064 & 0.305 & 0.385 & 0.245 \\ 0.051 & 0.292 & 0.172 & 0.172 & 0.232 \\ 0.418 & 0.056 & 0.296 & 0.055 & 0.417 \\ 0.167 & 0.288 & 0.047 & 0.127 & 0.047 \end{bmatrix}$$

$$G^{10} = \begin{bmatrix} 0.210 & 0.208 & 0.209 & 0.208 & 0.210 \\ 0.233 & 0.236 & 0.232 & 0.232 & 0.232 \\ 0.184 & 0.183 & 0.184 & 0.183 & 0.184 \\ 0.226 & 0.228 & 0.228 & 0.231 & 0.226 \\ 0.147 & 0.145 & 0.147 & 0.146 & 0.148 \end{bmatrix}$$

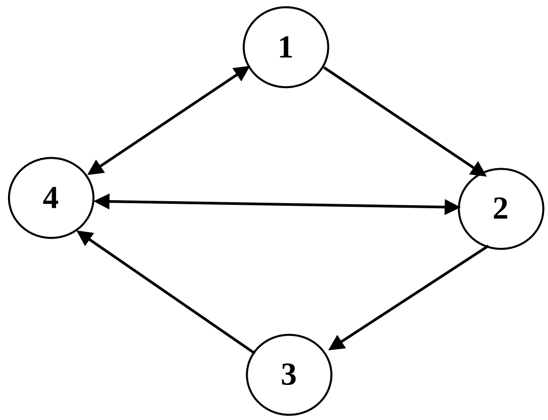
$$G^{1000} = \begin{bmatrix} 0.209 & 0.209 & 0.209 & 0.209 & 0.209 \\ 0.233 & 0.233 & 0.233 & 0.233 & 0.233 \\ 0.183 & 0.183 & 0.183 & 0.183 & 0.183 \\ 0.228 & 0.228 & 0.228 & 0.228 & 0.228 \\ 0.147 & 0.147 & 0.147 & 0.147 & 0.147 \end{bmatrix}$$

The vector $\begin{pmatrix} 0.209 \\ 0.233 \\ 0.183 \\ 0.228 \\ 0.147 \end{pmatrix}$ is the eigenvector of G corresponding to the eigenvalue $\lambda = 1$,

and determines the PageRank of the five webpages. Therefore, webpage 2 will be the highest rank, webpage 4 will be the 2nd highest, … , and webpage 5 will be the lowest rank webpage (more information during the lecture).

Homework

Consider the following network with the given links.



a. Construct the adjacency matrix (A) of the above graph.

b. Convert the adjacency matrix into a stochastic matrix (S).

c. Find all eigenvalues and eigenvectors of S.

d. What does the dominant eigenvector of S tell you about the network?

3. Towers of Hanoi